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LETTER TO THE EDITOR

***N*-level atom and (*N* - 1) modes: an exactly solvable model with detuning and multiphotons**

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Abstract. A model is presented for the interaction between an *N*-level atom and (*N* - 1) modes. Under the exact resonance condition between the modes with a detuning parameter, the Hamiltonian becomes solvable. The evolution operator is calculated and then the time development for the photon number operators and the occupation numbers are obtained. Multiphoton processes are discussed.

The interaction between electromagnetic fields and atoms lies at the heart of laser theory [1, 2], quantum optics [3, 4] some non-linear phenomena [5, 6] and laser spectroscopy [7, 8]. Some models have been presented to discuss different phenomena. The Jaynes-Cummings model [9] for the two-level atom has been investigated extensively to study emission, saturated absorption, the dynamic Stark effect [see 4, 8-11] and collapses and revivals [12-17].

The three-level atom model has been introduced to study stepwise and two-photon excitations [11, 18, 19], coherence trapping [20, 21] and two-photon lasers [22, 23]. The semiclassical treatment of the problem has also been given [20, 22, 23]. Recently the problem has been discussed in a full quantum mechanical manner to discuss dynamics of the system and collapses and revivals [14, 24-27]. The multilevel atom has been considered to treat the Dicke model [28] for a system of two-level atoms in interaction with radiation [3, 5]. It has been used recently to discuss multimode lasers [29].

In this letter we present a model for the interaction between a *N*-level atom and (*N* - 1) modes of radiation. This model includes a detuning parameter Δ and functions of the photon numbers in the modes. By specifying these functions we are capable of discussing multiphoton processes.

We assume an atom with *N* levels of energies $\omega_1 > \omega_2 > \dots > \omega_N$ interacts with (*N* - 1) modes whose energies are $\{\Omega_i\}$, $i = 1, 2, \dots, N - 1$. Each mode connects one of the lower energy levels to the upper level as in the scheme shown in figure 1.

According to this, the Hamiltonian of the system is written as follows in the rotating wave approximation [4]:

$$H = \sum_{i=1}^N \omega_i \hat{S}_{ii} + \sum_{i=1}^{N-1} \Omega_i \hat{a}_i^\dagger \hat{a}_i + \sum_{i=1}^{N-1} \lambda_i (\hat{S}_{1,i+1} \hat{R}_i + \hat{R}_i^\dagger \hat{S}_{i+1,1}) \tag{1}$$

where \hat{S}_{ij} are the generators of the $U(N)$ group [29] and \hat{R}_i is given by

$$\hat{R}_i = \hat{a}_i f_i(\{\hat{n}_i\}). \tag{2}$$

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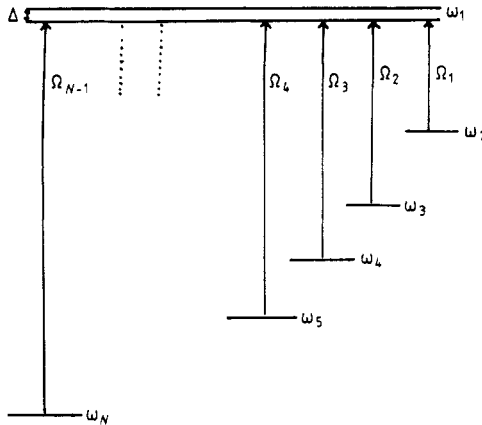


Figure 1. Schematic representation of the interaction between an N -level atom and $(N - 1)$ modes.

The operator \hat{a}_i (\hat{a}_i^+) is the annihilation (creation) boson operator for a photon in the field mode (i), $f_i(\{\hat{n}_i\})$ is a real function of the photons numbers in the modes and $\hat{n}_i = a_i^+ a_i$ is the number operator in the mode i . By specifying these functions later we will be able to discuss multiphoton processes. Finally $\{\lambda_i\}$ is the set of coupling constants between the atom and the modes.

We assume that \hat{S}_{ij} and \hat{R}_k commute, while we find the following commutation relations

$$\begin{aligned}
 [\hat{R}_j, \hat{n}_k] &= \hat{R}_j \delta_{jk} & [\hat{R}_j^+, \hat{n}_k] &= -\hat{R}_j^+ \delta_{jk} \\
 [\hat{S}_{ij}, \hat{S}_{kl}] &= \hat{S}_{il} \delta_{jk} - \hat{S}_{kj} \delta_{li}
 \end{aligned}
 \tag{3}$$

When we restrict the number N to 3 and put $f_i\{n_i\} = \text{constant}$ we get all models that have been discussed before for the three-level atom in the ‘ Λ ’ configuration [14, 24–27]. Singh’s model [16] for the two-level atom with intensity assisted transition is obtained when one substitutes $N = 2$, $f_i = f(n) = \sqrt{n}$. Of course, one obtains the standard Jaynes–Cummings model [9], when one further puts $f(n) = \text{constant}$. The model for the N -level atom of Li and Zhu [29] follows from (1) when one takes $f_i = \text{constant}$ and $\Delta = 0$.

Using the commutation relations (3) we find the following operators:

$$\hat{N}_j = \hat{a}_j^+ \hat{a}_j - \hat{S}_{j+1, j+1} \quad j = 1, 2, \dots, (N - 1)
 \tag{4}$$

to be constants of motion. This means that the difference between the photon number in the j th mode and the occupation number in the energy level ω_{j+1} is constant.

When we further use the condition

$$\omega_1 - \omega_{j+1} - \Omega_j = \Delta,
 \tag{5}$$

which is the resonance condition for the $(N - 1)$ modes, the model becomes exactly solvable and the Hamiltonian (1) breaks up into the following:

$$\hat{H} = \hat{D} + \hat{C}
 \tag{6}$$

where the two operators

$$\hat{D} = (\omega_1 - \Delta)I + \sum_{j=1}^{N-1} \Omega_j \hat{N}_j
 \tag{6a}$$

and

$$\hat{C} = \Delta \hat{S}_{11} + \sum_{j=1}^{N-1} \lambda_j (\hat{S}_{1,j+1} \hat{R}_j^+ + \hat{R}_j \hat{S}_{j+1,1}) \tag{6b}$$

commute, and hence are constants of motion.

We look now for the time evolution operator $\hat{U}(t)$ defined by

$$\hat{U}(t) = \exp(-i\hat{H}t) = \exp(-i\hat{D}t) \exp(-i\hat{C}t). \tag{7}$$

This form is due to the commutation of \hat{D} and \hat{C} . The term $\exp(-i\hat{D}t)$ introduces phase factors that would not affect the expectation values calculated later. The operator $\exp(-i\hat{C}t)$ is written in the following matrix form:

$$\exp(-i\hat{C}t) = \exp(-\frac{1}{2}i\Delta t) \begin{bmatrix} \hat{B}^{(0)} & \hat{B}^{(1)} \\ -\hat{B}^{(1)+} & \hat{B}^{(2)} \end{bmatrix} \tag{8}$$

where $\hat{B}^{(0)}$ is the single-element matrix

$$\{\hat{b}_{11}^{(0)} = \cos(\hat{\mu}t) - \frac{i}{2\mu} \sin(\hat{\mu}t)\} \tag{8a}$$

$\hat{B}^{(1)}$ is the single-row matrix

$$\{\hat{b}_{1j}^{(1)} = -i \sin(\hat{\mu}t) \lambda_j \hat{R}_j\} \quad j \in \{1, \dots, N-1\} \tag{8b}$$

and $\hat{B}^{(1)}$ its Hermitian conjugate, while $\hat{B}^{(2)}$ is the $(N-1) \times (N-1)$ square matrix

$$\{\hat{b}_{ij}^{(2)} = \exp(\frac{1}{2}i\Delta t) \delta_{ij} + \lambda_i \hat{R}_i^+ \hat{A} \lambda_j \hat{R}_j\} \quad i, j \in \{1, \dots, N-1\} \tag{8c}$$

the $\hat{\nu}$, $\hat{\mu}$ and \hat{A} operators are given by:

$$\nu = \sum_{j=1}^{N-1} \lambda_j \hat{R}_j \hat{R}_j^+ = \sum_{j=1}^{N-1} \lambda_j^2 (\hat{n}_j + 1) f_j^2(\hat{n}_1, \dots, \hat{n}_j + 1, \dots, \hat{n}_{N-1}) \tag{9a}$$

$$\hat{\mu}^2 = \hat{\nu} + \Delta^2/4 \quad \hat{A} = \left\{ \cos(\hat{\mu}t) - \exp(\frac{1}{2}i\Delta t) + \frac{i\Delta}{2\mu} \sin(\hat{\mu}t) \right\} \hat{\nu}^{-1}. \tag{9b}$$

They satisfy

$$\hat{a}_j^+ \hat{\nu} = \hat{\nu}_j \hat{a}_j^+$$

where

$$\hat{\nu}_j = \lambda_j^2 \hat{n}_j f_j^2(\{\hat{n}_i\}) + \sum_{\substack{(k \neq j) \\ k=1}} \lambda_k^2 (\hat{n}_k + 1) f_k^2(\hat{n}_1, \dots, \hat{n}_j - 1, \hat{n}_k + 1, \dots, \hat{n}_{N-1}). \tag{9c}$$

the following operators are defined:

$$\hat{\mu}_j^2 = \hat{\nu}_j + \Delta^2/4 \quad \hat{A}_j = \left\{ \cos(\hat{\mu}_j t) - \exp(\frac{1}{2}i\Delta t) + \frac{i\Delta}{2\mu_j} \sin(\hat{\mu}_j t) \right\} \hat{\nu}_j^{-1}. \tag{9d}$$

When we take $\Delta = 0$ and $f_j = 1$, we get the results of [29].

The quantities $\hat{\mu}$ and $\hat{\mu}_j$ are Rabi frequencies in this case. They are direct generalisations to Rabi frequencies in the two- and three-level atom systems [9, 14-16, 24-27].

Once the operator $\hat{U}(t)$ is calculated, we can compute the time development for any operator \hat{O} through the equation

$$\hat{O}(t) = \hat{U}(t)\hat{O}\hat{U}^+(t). \quad (10)$$

One of the quantities on which we shall concentrate is the probability distribution function $P(\{n_j\}, t)$ of finding n_i photons in the j th mode, which is defined as follows:

$$P(\{n_j\}, t) = \langle \{n_j\} | \text{Tr}_A \hat{U}(t) \hat{\rho}(0) \hat{U}^+(t) | \{n_j\} \rangle. \quad (11)$$

The trace over the atomic states of the density matrix operator $\hat{\rho}$ of the system gives the density matrix operator for the fields $\hat{\rho}_F$.

We consider in what follows the system to be initially in one of its pure states.

(a) Atom is initially in one of its states of energy ω_{k+1} ($k \neq 0$). The initial density matrix operator is given by

$$\rho^k(0) = \rho_F(0) \otimes \hat{S}_{k+1, k+1}. \quad (12a)$$

The distribution function takes the form

$$\begin{aligned} P_{\Delta}^k(\{n_i\}, t) &= \lambda_k^2(n_k + 1) f_k^2(n_1, \dots, n_k + 1, \dots, n_{N-1}) \frac{\sin^2(\mu t)}{\mu^2} \\ &\quad \times P(n_1, \dots, n_k + 1, \dots, n_{N-1}) + |\exp(\frac{1}{2}i\Delta t) + A_k \lambda_k^2 n_k f_k^2(\{n_i\})|^2 P(\{n_i\}) \\ &\quad + \sum_{j \neq k} \lambda_k^2(n_k + 1) f_k^2(n_1, \dots, n_j - 1, \dots, n_k + 1, \dots, n_{N-1}) \lambda_j^2 n_j \\ &\quad \times f_j^2(\{n_i\}) |A_j|^2 P(n_1, \dots, n_j - 1, \dots, n_k + 1, \dots, n_{N-1}) \end{aligned} \quad (12b)$$

where $P(\{n_j\})$ is the initial value for the probability distribution function.

The expectation value for the photon numbers are given by

$$\begin{aligned} \langle \hat{n}_j(t) \rangle_{\Delta}^k &= \bar{n}_j + \sum_{\{n_i\}} P(\{n_i\}) \lambda_j^2(n_j + 1) f_j^2(n_1, \dots, n_j + 1, n_k - 1, \dots, n_{N-1}) \\ &\quad \times \lambda_k^2 n_k f_k^2(\{n_i\}) |A_k|^2 \quad (j \neq k) \end{aligned} \quad (13a)$$

while

$$\langle \hat{n}_k(t) \rangle_{\Delta}^k = \bar{n}_k - \sum_{\{n_j\}} P(\{n_j\}) \lambda_k^2 n_k f_k^2(\{n_j\}) \left(\frac{\sin^2(\mu_k t)}{\mu_k^2} + (\nu_k - \lambda_k^2 n_k f_k^2(\{n_j\}) |A_k|^2) \right). \quad (13b)$$

The expectation values for the occupation numbers are given in this case by

$$\langle \hat{S}_{j+1, j+1}(t) \rangle_{\Delta}^k = \langle \hat{n}_j(t) \rangle_{\Delta}^k - \bar{n}_j \quad \text{for } j \neq k \quad (14a)$$

$$\langle \hat{S}_{k+1, k+1}(t) \rangle_{\Delta}^k = \langle \hat{n}_k(t) \rangle_{\Delta}^k - \bar{n}_k + 1 \quad (14b)$$

while we find

$$\langle \hat{S}_{11}(t) \rangle_{\Delta}^k = \sum_{\{n_j\}} P(\{n_j\}) \lambda_k^2 n_k f_k^2(\{n_j\}) \frac{\sin^2(\mu_k t)}{\mu_k^2}. \quad (14c)$$

When $\Delta = 0$ and $f = 1$ we get the results of [29], while the results of the 'A' configuration for the three-level atom [25-27] are recovered when we write $N = 3$ and $f = 1$.

(b) Atom is initially in its upper state of energy ω_1 . In this case the initial density operator is given by

$$\hat{\rho}^u(0) = \hat{\rho}_F(0) \otimes \hat{S}_{11} \quad (15a)$$

and the probability distribution function for the photons is

$$P_{\Delta}^u(\{n_j\}, t) = \left(\cos^2(\mu t) + \frac{\Delta^2}{4\mu^2} \sin^2(\mu t) \right) P(\{n_j\}) + \sum_{k=1}^{N-1} \lambda_k^2 n_k f_k^2(\{n_j\}) \frac{\sin^2(\mu_k t)}{\mu_k^2} P(n_1, \dots, n_k - 1, \dots, n_{N-1}). \tag{15b}$$

The photon number dependence on time is given by

$$\langle \hat{n}_j(t) \rangle_{\Delta}^u = \bar{n}_j + \sum_{\{n_i\}} \lambda_j^2 (n_j + 1) f_j^2(n_1, \dots, n_j + 1, \dots, n_{N-1}) \frac{\sin^2(\mu t)}{\mu^2} P(\{n_i\}) \tag{16a}$$

while

$$\langle \hat{S}_{j+1, j+1}(t) \rangle_{\Delta}^u = \langle \hat{n}_j(t) \rangle_{\Delta}^u - \bar{n}_j \quad j \neq N \tag{16b}$$

and

$$\langle \hat{S}_{11}(t) \rangle_{\Delta}^u = \sum_{\{n_i\}} \left(\cos^2(\mu t) + \frac{\Delta^2}{4\mu^2} \sin^2(\mu t) \right) P(\{n_i\}). \tag{16c}$$

It is apparent that the oscillations in these quantities depend on the Rabi frequency 2μ . It is observed that the system develops in time in this case even when all modes start from vacuum.

We now look at the multiphoton processes. If we assume that the transition between the two levels ω_1 and ω_{j+1} is effected through m_j photons of the j th mode, then the Hamiltonian is written as

$$H = \sum_{i=1}^N \omega_i \hat{S}_{ii} + \sum_{i=1}^{N-1} \Omega_i \hat{a}_i^{\dagger} \hat{a}_i + \sum_{i=1}^{N-1} \lambda_i \{ \hat{S}_{1, i+1} \hat{a}_i^{m_i} + \hat{a}_i^{+m_i} \hat{S}_{i+1, 1} \} \tag{17}$$

when we use the generalised boson operators \hat{b}_i such that

$$\hat{a}_i^{m_i} = \hat{b}_i \left(\frac{\hat{n}_i!}{[\hat{n}_i/m_i] (\hat{n}_i - m_i)!} \right)^{1/2} = \hat{b}_i f_i(\hat{n}_i) \tag{18a}$$

where the relation between $\hat{n}_{b_i} = \hat{b}_i^{\dagger} \hat{b}_i$ and $\hat{n}_i = \hat{a}_i^{\dagger} \hat{a}_i$ is given by [30]

$$\hat{n}_i = m_i \hat{n}_{b_i}. \tag{18b}$$

The Hamiltonian (19) now takes the form

$$H = \sum \omega_i \hat{S}_{ii} + \sum m_i \Omega_i \hat{b}_i^{\dagger} \hat{b}_i + \sum \lambda_i \{ \hat{S}_{1, i+1} \hat{b}_i f_i(\hat{n}_i) + f_i(\hat{n}_i) \hat{b}_i^{\dagger} \hat{S}_{i+1, 1} \} \tag{19}$$

which is the same form as (1), but the function f_i is a function of n_i only and given by (18a). The ν of (9) are given by

$$\nu = \sum_k \lambda_k^2 \frac{(n_k + m_k)!}{n_k!} \quad \nu_j = \lambda_j^2 \left(\frac{n_j!}{(n_j - m_j)!} \right) + \sum_{k \neq j} \lambda_k^2 \frac{(n_k + m_k)!}{n_k!}. \tag{20}$$

For example, we find that $\langle \hat{n}_j(t) \rangle^k$ of (13a) is given by

$$\langle \hat{n}_j(t) \rangle_{\text{multi}}^k = \bar{n}_j + \sum_{\{n_i\}} P(n_1, \dots, n_{N-1}) \lambda_j^2 \lambda_k^2 \left(\frac{(n_j + m_j)!}{n_j!} \right) \left(\frac{n_k!}{(n_k - m_k)!} \right) |A_k|^2 m_j \tag{21}$$

where A_k and μ_k are given by (9d) with ν_k given by (20).

We have presented a discussion of the model of an N -level atom and $(N - 1)$ modes. Constants of motion are obtained. Under condition (5) the Hamiltonian becomes solvable. The photon number operators are computed when the atom is initially in one of its states. The occupation numbers in the levels are also given. The dependence on the detuning parameter Δ is shown.

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